B.Sc. Part I (Hons.) 2nd Paper

Diff. Calculus

Successive Differentiation (contd.)

Q. If $\cos^{-1}\frac{y}{b} = \log\left(\frac{x}{n}\right)^n$, prove that

$x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0.$

Soln Given that $\cos^{-1}\frac{y}{b} = \log\left(\frac{x}{n}\right)^n$

$\Rightarrow \cos^{-1}\frac{y}{b} = n \log\frac{x}{n}$

Differentiating with respect to x, we get

$\Rightarrow \frac{-1}{\sqrt{1-\frac{y^2}{b^2}}} \cdot \frac{1}{b} \cdot y_1 = \cancel{n} \cdot \frac{1}{\frac{x}{n}} \times \frac{1}{\cancel{x}}$

$\Rightarrow \frac{-\cancel{b}}{\cancel{b}\sqrt{b^2-y^2}} \cdot y_1 = \frac{n}{x}$

$\Rightarrow x y_1 = -n\sqrt{b^2-y^2}$. squaring both sides, we've

$\Rightarrow n^2(b^2-y^2) = x^2 y_1^2$

Differentiating with respect to x, we get

$\Rightarrow -2n^2 \cdot y y_1 = 2x y_1^2 + x^2 \cdot 2y_1 y_2$

$\Rightarrow 2y_1(-n^2 y) = 2y_1(x^2 y_2 + x y_1)$

$\Rightarrow x^2 y_2 + x y_1 + n^2 y = 0$

Differentiating n times by Leibnitz's theorem, we get

$$\left[y_2 x^2\right]_n + \left[y_1 x\right]_n + n^2\left[y\right]_n = 0$$

$$\Rightarrow y_{n+2}\, x^2 + (y_2)_{n-1}\, {}^nc_1 \left[x^2\right]_1 + {}^nc_2 \left[y_2\right]_{n-2} \left[x^2\right]_2$$

$$+ \left[y_1\right]_n x + {}^nc_1 \left[y_1\right]_{n-1} \left[x\right]_1 + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + n\, y_{n+1} \cdot 2x + \frac{n(n-1)}{\cancel{2}} \cdot y_n \cdot \cancel{2}$$

$$+ y_{n+1}\, x + n\, y_n \cdot 1 + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + 2nx\, y_{n+1} + n(n-1) y_n$$

$$+ x y_{n+1} + n y_n + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\, y_{n+1} + y_n\left[n(n-1) + n + n^2\right] = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2 - \cancel{n} + \cancel{n} + n^2) y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0$$